ENSTROPHY AND ERGODICITY OF GRAVITY CURRENTS

VENA PEARL BONGOLAN-WALSH, JINQIAO DUAN, HONGJUN GAO,
TAMAY ÖZGÖKMEN, PAUL FISCHER, AND TRAIAN ILIESCU

ABSTRACT. We study a coupled deterministic system of vorticity evolution
and salinity transport equations, with spatially correlated white noise on the
boundary. This system may be considered as a model for gravity currents
in oceanic fluids. The noise is due to uncertainty in salinity flux on fluid
boundary. After transforming this system into a random dynamical system,
we first obtain an asymptotic estimate of enstrophy evolution, and then show
that the system is ergodic under suitable conditions on mean salinity input
flux on the boundary, Prandtl number and covariance of the noise.

1. GEOPHYSICAL BACKGROUND

A gravity current is the flow of one fluid within another driven by the gravitational
force acting on the density difference between the fluids. Gravity currents occur in
a wide variety of geophysical fluids.

Oceanic gravity currents are of particular importance, as they are intimately related
to the ocean's role in climate dynamics. The thermohaline circulation in the ocean
is strongly influenced by dense-water formation that takes place mainly in polar
seas by cooling and in marginal seas by evaporation. Such dense water masses are
released into the large-scale ocean circulation in the form of overflows, which are
bottom gravity currents.

We consider a two-dimensional model for oceanic gravity currents, in terms of the
Navier-Stokes equation in vorticity form and the transport equation for salinity.
The Neumann boundary conditions for this model involve a spatially correlated
white noise due to uncertain salinity flux at the inlet boundary of the gravity
currents.

In the next section, we present the model and reformulate it as a random dynam-
ical system, and then discuss the cocycle property and dissipativity of this model
in §3 and §4, respectively. Main results on random attractors, enstrophy and er-
godicity are in §5. Enstrophy is one half of the mean-square spatial integral of
vorticity. Ergodicity implies that the time average for observables of the dynamical
system approximates the statistical ensemble average, as long as the time interval
is sufficiently long.

2. MATHEMATICAL MODEL

Oceanic gravity currents are usually down a slope of small angle (order of a few
degrees). We model the gravity currents in the downstream-vertical plane, and

ignore the variability in the cross-stream direction. This is an appropriate approximate model for, e.g., the Red Sea overflow that flows along a long narrow channel that naturally restricts motion in the lateral planar plane [11]. In fact, we will ignore small slope angle and the rotation, both affect the following estimates non-essentially, i.e., our results below still hold with non-essential modification of constants in the estimates and in the conditions for the ergodicity.

Thus we consider the gravity currents in the downstream-vertical (x, z)-plane. It is composed of the Boussinesq equations for ocean fluid dynamics in terms of vorticity $q(x, z, t)$, and the transport equation for oceanic salinity $S(x, z, t)$ on the domain $D = \{(x, z) : 0 \leq x, z \leq 1\}$:

(1)
$$q_t + J(q, \psi) = \Delta q - Ra\partial_x S,$$
$$S_t + J(S, \psi) = \frac{1}{\mathrm{Pr}}\Delta S,$$

where

$$q(x, z, t) = -\Delta\psi$$

is the vorticity in terms of stream function ψ, Pr is the Prandtl number and Ra is the Rayleigh number. Moreover, $J(g, h) = g_x h_z - g_z h_x$ is the Jacobian operator and $\Delta = \partial_{xx} + \partial_{zz}$ is the Laplacian operator. All these equations are in non-dimensionalized forms. For the simplicity, we let $Pr = 1$.

Note that the Laplacian operator Δ in the temperature and salinity transport equations is presumably $\partial_{xx} + \frac{\kappa_V}{\kappa_H}\delta^2\partial_{zz}$ with δ being the aspect ratio, and κ_H, κ_V the horizontal and vertical diffusivities of salt, respectively. However, our energy-type estimates and the results below will not be essentially affected by taking a homogenized Laplacian operator $\Delta = \partial_{xx} + \partial_{zz}$. All our results would be true for this modified Laplacian. The effect of the rotation is parameterized in the magnitude of the viscosity and diffusivity terms as discussed in [19].

The fluid boundary condition is no normal flow and free-slip on the whole boundary

$$\psi = 0, \ q = 0.$$

The flux boundary conditions are assumed for the ocean salinity S.

At the inlet boundary $\{x = 0, \ 0 < z < 1\}$ the flux is specified as:

(2)
$$\partial_x S = F(z) + \dot{w}(z, t),$$

with $F(z)$ being the mean freshwater flux, and the fluctuating part $\dot{w}(z, t)$ is usually of a shorter time scale than the response time scale of the oceanic mean salinity. So we neglect the autocorrelation time of this fluctuating process and thus assume that the noise is white in time. The spatially correlated white-in-time noise $\dot{w}(z, t)$ is described as the generalized time derivative of a Wiener process $w(z, t)$ defined in a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, with mean vector zero and covariance operator Q. On the outlet boundary $\{x = 1, \ 0 < z < 1\}$:

$$\partial_x S = 0.$$

At the top boundary $z = 1$, and at the bottom boundary $z = 0$:

$$\partial_z S = 0 .$$

This is a system of deterministic partial differential equations with a stochastic boundary condition.

3. COCYCLE PROPERTY

In this section we will show that (1) has a unique solution, and by reformulating the model, we see it defines a cocycle or a random dynamical system.

For the following we need some tools from the theory of partial differential equations.

Let $W_2^1(D)$ be the Sobolev space of functions on D with first generalized derivative in $L_2(D)$, the function space of square integrable functions on D with norm and inner product

$$\|u\|_{L_2} = \left(\int_D |u(x)|^2 dD \right)^{\frac{1}{2}}, \quad (u,v)_{L_2} = \int_D u(x)v(x)dD, \quad u,\, v \in L_2(D).$$

The space $W_2^1(D)$ is equipped with the norm

$$\|u\|_{W_2^1} = \|u\|_{L_2} + \|\partial_x u\|_{L_2} + \|\partial_z u\|_{L_2}.$$

Motivated by the zero-boundary conditions of q we also introduce the space $\overset{\circ}{W}_2^1(D)$ which contains roughly speaking functions which are zero on the boundary ∂D of D. This space can be equipped with the norm

$$\|u\|_{\overset{\circ}{W}_2^1(D)} = \|\partial_x u\|_{L_2} + \|\partial_z u\|_{L_2}. \tag{3}$$

Similarly, we can define function spaces on the interval $(0,1)$ denoted by $L_2(0,1)$. Another Sobolev space is given by $\dot{W}_2^1(D)$ which is a subspace of $W_2^1(D)$ consisting of functions h such that $\int_D h\,dD = 0$. A norm equivalent to the W_2^1-norm on $\dot{W}_2^1(D)$ is given by the right hand side of (3). For the subspace of functions in $L_2(D)$ having this property we will denote as $\dot{L}_2(D)$.

We reformulate the above stochastic initial-boundary value problem into a random dynamical system [1]. For convenience, we introduce vector notation for unknown geophysical quantities

$$u = (q, S). \tag{4}$$

Let $\dot{w}$ be a white noise in $L_2(0,1)$ with finite trace of the covariance operator Q, and the Wiener process $w(t)$ be defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$.

Now we choose an appropriate phase space H for this system. We assume that the mean salinity flux $F \in L_2(0,1)$. Note that

$$\frac{d}{dt} \int_D S\,dD = \int_0^1 [F(z) + \dot{w}(z,t)]dz = \text{constant}.$$

It is reasonable (see [5]) to assume that

$$\int_0^1 [F(z) + \dot{w}(z,t)]dz = 0, \tag{5}$$

and thus $\int_D S\,dD$ is constant in time and, without loss of generality, we may assume it is zero (otherwise, we subtract this constant from S):

$$\int_D S\,dD = 0.$$

Thus $S \in \dot{L}_2(D)$, and we have the usual Poincaré inequality for S.

Define the phase space

$$H = L_2(D) \times \dot{L}_2(D).$$

We rewrite the coupled system (1) as:

(6) $$\frac{du}{dt} + Au = F_1(u) + F_2(u), \qquad u(0) = u_0 \in H,$$

where

$$Au = \begin{pmatrix} -\Delta q \\ -\frac{1}{\mathrm{Pr}}\Delta S \end{pmatrix},$$

$$F_1(u)[x, z] = \begin{pmatrix} -J(q, \psi) \\ -J(S, \psi) \end{pmatrix} [x, z],$$

and

$$F_2(u)[x, z] = \begin{pmatrix} \mathrm{Ra}(-\partial_x S) \\ 0 \end{pmatrix} [x, z].$$

The boundary and initial conditions are:

(7)
$$q = 0, \text{on } \partial D,$$
$$\frac{\partial S}{\partial n} = 0, \text{ on } \partial D \backslash \{x = 0, \ 0 < z < 1\},$$
$$\frac{\partial S}{\partial x} = F(z) + \dot{w}(z, t), \text{ on } \{x = 0, \ 0 < z < 1\},$$
$$u(0) = u_0 = \begin{pmatrix} q_o \\ S_o \end{pmatrix},$$

where n is the out unit normal vector of ∂D. The system (6) consists of deterministic partial differential equations with stochastic Neumann boundary conditions.

We now transform the above system (6) into a system of random partial differential equations (i.e., evolution equations with random coefficients) with homogeneous boundary conditions, whose solution map can be easily seen as a cocycle. Thus we can investigate this dynamics in the framework of random dynamical systems [1]. Note that we have a non-homogenous stochastic boundary condition for salinity S, so the first step is to homogenize this boundary condition.

To this end, we need an Ornstein-Uhlenbeck stochastic process solving the linear differential equation

(8) $$\frac{d\eta_1}{dt} = \Delta\eta_1,$$

with the following same boundary conditions as for S, and zero initial condition

(9)
$$
\begin{aligned}
\partial_x \eta_1(t, 0, z, \omega) &= F(z) + \dot{w}(z, t), \\
\partial_x \eta_1(t, 1, z, \omega) &= 0, \\
\partial_z \eta_1(t, x, 0, \omega) &= 0, \\
\partial_z \eta_1(t, x, 1, \omega) &= 0, \\
\eta_1(0, x, z, \omega) &= 0, \\
\int_D \eta_1 \, dD &= 0.
\end{aligned}
$$

Lemma 3.1. *Suppose that the covariance Q has finite trace : $\mathrm{tr}_{L_2} Q < \infty$. Then the Ornstein-Uhlenbeck problem (8)-(9) has a unique stationary solution in $L_2(D)$ generated by*

$$(t, \omega) \rightarrow \eta_1(\theta_t \omega).$$

In fact, we can write down the solution η_1 following Da Prato and Zabczyck [13, 12] as

(10)
$$
\eta_1(t, x, z, \omega) = (-\Delta) \int_0^t S(t - s) \mathcal{N}(\mathbf{X}) \, ds
$$

where I is the identity operator in $L_2(D)$, and $\mathcal{N}$ is the solution operator to the elliptic boundary value problem $\Delta h - \lambda h = 0$ with the boundary conditions for h the same as η_1, that is $\partial h / \partial n = \mathbf{X}$ on ∂D with $\int_D h \, dD = 0$, where n is the unit outer normal vector to ∂D and $\mathbf{X}$ is

$$
\mathbf{X} = \begin{pmatrix} F(z) + \dot{w}(z, s) \\ 0 \\ 0 \\ 0 \end{pmatrix}.
$$

Here λ is chosen so that this elliptic boundary value problem has a unique solution. Since $\int_D h \, dD = 0$, we can choose $\lambda = 0$. Moreover, S(t) is a strongly continuous semigroup, symbolically, $e^{\Delta t}$, that is, the generator of $S(t)$ is Δ.

Now we are ready to transform (6) into a random dynamical system in Hilbert space H. Define

(11)
$$
\eta(t, x, z, \omega) = \begin{pmatrix} 0 \\ \eta_1 \end{pmatrix}
$$

and recall

$$
u = \begin{pmatrix} q \\ S \end{pmatrix}.
$$

Let

(12)
$$
v := u - \eta.
$$

Then we obtain a random partial differential equation

(13)
$$
\frac{dv}{dt} + Av = F_1(v + \eta(\theta_t \omega)) + F_2(v + \eta(\theta_t \omega)), \qquad v(0) = v_0 \in H.
$$

where

$v_x(t, 0, z, \omega)$, $v_x(t, 1, z, \omega)$, $v_z(t, x, 0, \omega)$ and $v_z(t, x, 1, \omega)$ are now all zero vectors,i.e., homogenous boundary conditions, and the initial condition is still the same as for u

$$v(0, x, z, \omega) = \begin{pmatrix} q_o \\ S_o \end{pmatrix},$$

However, because of the Jacobian, we will have to do nonlinear analysis on (13) to resolve v.

We introduce another space

$$V = \overset{\circ}{W}{}^1_2(D) \times \dot{W}^1_2(D)$$

.

For sufficiently smooth functions $v = (\bar{q}, \bar{S})$, we can calculate via integration by parts

(14)
$$\begin{aligned} (\mathcal{A}u, v)_H &= \int_D \nabla q \cdot \nabla \bar{q} \, dD \\ &+ \frac{1}{Pr} \int_D \nabla S \cdot \nabla \bar{S} \, dD \end{aligned}$$

since now, we have only homogenous boundary conditions.

Hence on the space V we can introduce a bilinear form $\tilde{a}(\cdot, \cdot)$ which is continuous, symmetric and positive

$$\tilde{a}(u, v) = \int_D \nabla q \cdot \nabla \bar{q} dD + \int_D \nabla S \cdot \nabla \bar{S} dD$$

This bilinear form defines a unique linear continuous operator $A : V \to V'$ such that $\langle Au, v \rangle = \tilde{a}(u, v)$.

Recall

$$F_1(u)[x, z] = \begin{pmatrix} -J(q, \psi) \\ -J(S, \psi) \end{pmatrix} [x, z].$$

and

$$F_2(u)[x, z] = \begin{pmatrix} \mathrm{Ra}(-\partial_x S) \\ 0 \end{pmatrix} [x, z].$$

Lemma 3.2. *The operator $F_1 : V \to H$ is continuous. In particular, we have*

$$\langle F_1(u), u \rangle = 0.$$

Proof. We have a constant $c_1 > 0$ such that

(15)
$$\|\psi\|_{W^3_2(D)} \leq c_1 \|q\|_{W^1_2(D)}$$

for any $q \in W_2^1(D)$ which follows straight forwardly by regularity properties of a linear elliptic boundary problem. Note that W_2^3 is a Sobolev space with respect to the third derivatives. Hence we get:

$$\|J(S, \psi)\|_{L_2} \leq \sup_{(x,z) \in D} (|\partial_x \psi(x,z)| + |\partial_z \psi(x,z)|) \times$$

$$\times \left(\int_D |\partial_x S(x,z)| + |\partial_z S(x,z)| dD \right).$$

The second factor on the right hand side is bounded by

$$\left(\int_D |\partial_x S(x,z)|^2 dD \right)^{\frac{1}{2}} + \left(\int_D |\partial_z S(x,z)|^2 dD \right)^{\frac{1}{2}} \leq \|u\|_V.$$

On account of the Sobolev embedding Lemma, we have some positive constants c_2, c_3 such that

$$\sup_{(x,z) \in D} (|\partial_x \psi(x,z)| + |\partial_z \psi(x,z)|) \leq c_2 \|\nabla \psi\|_{W_2^2(D)} \leq c_3 \|q\|_{W_2^1(D)} \leq c_3 \|u\|_V.$$

Hence we have a positive constant c_4 such that

$$\|J(S, \psi)\|_{L_2} \leq c_4 \|u\|_V^2$$

for $u \in V$.

We now show that

$$\langle J(S, \psi), S \rangle = 0.$$

We obtain via integration by parts

$$\int_D \partial_x S \partial_z \psi \, SdD - \int_D \partial_z S \partial_x \psi \, SdD$$

$$= - \int_D \partial_{xz}^2 S \psi \, SdD + \int_D \partial_{zx}^2 S \psi S \, dD - \int_D \partial_x S \psi \partial_z S \, dD + \int_D \partial_z S \psi \partial_x S \, dD$$

$$+ \int_{(0,1)} \partial_x S \psi S|_{z=0}^{z=1} dx - \int_{(0,1)} \partial_z S \psi S|_{x=0}^{x=1} dz = 0$$

because ψ is zero on the boundary ∂D. This relation is true for a set of sufficiently smooth functions ψ, S which are dense in $\overset{\circ}{W_2^1}(D) \times W_2^1(D)$. By the continuity of F_1, as just shown in Lemma 3.2, we can extend this property to $\overset{\circ}{W_2^1}(D) \times W_2^1(D)$. $\square$

Lemma 3.3. *The following estimate holds*

$$\|F_2(u)\|_{L^2} \leq c_5 \|u\|_V.$$

for some positive constant c_5.

Proof. By simple calculation, the proof is obtained. $\square$

We have obtained a differential equation without white noise but with random coefficients. Such a differential equation can be treated sample-wise for *any* sample ω. We are looking for solutions in

$$v \in C([0, \tau]; H) \cap L^2(0, \tau; V),$$

for all $\tau > 0$. If we can solve this equation then $u := v + \eta$ defines a solution version of (6).

For the well posedness of the problem we now have the following result.

Theorem 3.4. (**Well-Posedness**) *For any time $\tau > 0$, there exists a unique solution of (13) in $C([0, \tau]; H) \cap L_2(0, \tau; V)$. In particular, the solution mapping*

$$\mathbb{R}^+ \times \Omega \times H \ni (t, \omega, v_0) \to v(t) \in H$$

is measurable in its arguments and the solution mapping $H \ni v_0 \to v(t) \in H$ is continuous.

Proof. By the properties of A and F_1 (see Lemma 3.2), the random differential equation (13) is essentially similar to the 2 dimensional Navier Stokes equation. Note that F_2 is only an affine mapping. Hence we have existence and uniqueness and the above regularity assertions. □

On account of the transformation (12), we find that (6) also has a unique solution. Since the solution mapping

$$\mathbb{R}^+ \times \Omega \times H \ni (t, \omega, v_0) \to v(t, \omega, v_0) =: \varphi(t, \omega, v_0) \in H$$

is well defined, we can introduce a random dynamical system. On Ω we can define a shift operator θ_t on the paths of the Wiener process that pushes our noise:

$$w(\cdot, \theta_t \omega) = w(\cdot + t, \omega) - w(t, \omega) \quad \text{for } t \in \mathbb{R}$$

which is called the *Wiener shift*. Then $\{\theta_t\}_{t \in \mathbb{R}}$ forms a flow which is ergodic for the probability measure $\mathbb{P}$. The properties of the solution mapping cause the following relations

$$\varphi(t + \tau, \omega, u) = \varphi(t, \theta_\tau \omega, \varphi(\tau, \omega, u)) \quad \text{for } t, \tau \geq 0$$

$$\varphi(0, \omega, u) = u$$

for any $\omega \in \Omega$ and $u \in H$. This property is called the cocycle property of φ which is important to study the dynamics of random systems. It is a generalization of the semigroup property. The cocycle φ together with the flow θ forms a *random dynamical system*.

4. Dissipativity

In this section we show that the random dynamical system (13) for gravity currents is dissipative, in the sense that it has an absorbing (random) set. This means that the solution v is contained in a particular region of the phase space H after a sufficiently long time. This dissipativity will help us to obtain asymptotic estimates of the enstrophy and salinity evolution. Dynamical properties that follow from this dissipativity will be considered in the next section. In particular, we will show that the system has a random attractor, and is ergodic under suitable conditions.

We introduce the spaces

$$\tilde{H} = \dot{L}_2(D)$$

$$\tilde{V} = \dot{W}_2^1(D).$$

We also choose a subset of dynamical variables of our system (1).

(16) $$\tilde{v} = S - \eta_1.$$

To calculate the energy inequality for $\tilde{v}$, we apply the chain rule to $\|\tilde{v}\|_H^2$. We obtain by Lemma 3.2

$$
(17) \qquad \begin{aligned} &\frac{d}{dt}\|\tilde{v}\|_{\tilde{H}}^2 + 2\|\nabla\tilde{v}\|_{L_2}^2 \\ &= 2\langle J(\eta_1,\psi),\tilde{v}\rangle. \end{aligned}
$$

The expression $\nabla\tilde{v}$ is defined by $(\nabla_{x,z}\tilde{v})$. We now can estimate the term on the right hand side.

By the Cauchy inequality, integration by parts and Poincare inequality $\lambda_1\|q\|_{L_2} \leq \|\nabla q\|_{L_2}$ for $q \in \overset{\circ}{W}{}_2^1(D)$ and $\lambda_2\|\tilde{v}\|_{L_2} \leq \|\nabla\tilde{v}\|_{L_2}$ for $\tilde{v} \in \dot{W}_2^1(D)$, we have

$$
(18) \qquad \frac{d}{dt}\|\tilde{v}\|_{\tilde{H}}^2 + \|\nabla\tilde{v}\|_{L_2}^2 \leq \lambda_1^2\|\nabla q\|^2\|\eta_1\|^2.
$$

For q, we have the following estimate

$$
(19) \qquad \frac{d}{dt}\|q\|^2 + \|\nabla q\|^2 \leq Ra^2(\lambda_2^2\|\nabla\tilde{v}\|^2 + \|\eta_1\|^2).
$$

From (18) and (19), we have

$$
(20) \quad \frac{d}{dt}\left(2\|\tilde{v}\|_{\tilde{H}}^2 + \frac{1}{Ra^2\lambda_2^2}\|q\|^2\right) + \|\nabla\tilde{v}\|_{L_2}^2 + \left(\frac{1}{Ra^2\lambda_2^2} - 2\lambda_1^2\|\eta_1\|^2\right)\|\nabla q\|^2 \leq \frac{1}{\lambda_2^2}\|\eta_1\|^2.
$$

Definition 4.1. *A random set $B = \{B(\omega)\}_{\omega\in\Omega}$ consisting of closed bounded sets $B(\omega)$ is called absorbing for a random dynamical system φ if we have for any random set $D = \{D(\omega)\}_{\omega\in\Omega}$, $D(\omega) \in H$ bounded, such that $t \to \sup_{y\in D(\theta_t\omega)}\|y\|_H$ has a subexponential growth for $t \to \pm\infty$*

$$
(21) \qquad \begin{aligned} \varphi(t,\omega,D(\omega)) &\subset B(\theta_t\omega) \quad \text{for } t \geq t_0(D,\omega) \\ \varphi(t,\theta_{-t}\omega,D(\theta_{-t}\omega)) &\subset B(\omega) \quad \text{for } t \geq t_0(D,\omega). \end{aligned}
$$

B is called forward invariant if

$$
\varphi(t,\omega,u_0) \in B(\theta_t\omega) \quad \text{if } u_0 \in B(\omega) \quad \text{for } t \geq 0.
$$

Although $\tilde{v}$ is not a random dynamical system in the strong sense we can also show dissipativity in the sense of the above definition.

Lemma 4.2. *Let $\varphi(t,\omega,v_0) \in H$ for $v_0 \in H$ be defined in (6), and*

$$
\frac{1}{Ra^2\lambda_2^2} - 2\lambda_1^2\mathbb{E}\|\eta_1\|^2 > 0.
$$

Then the closed ball $B(0,R_1(\omega))$ with radius

$$
R_1(\omega) = 2\int_{-\infty}^0 e^{\alpha\tau}\frac{1}{\lambda_2^2}\|\eta_1\|^2 d\tau
$$

is forward invariant and absorbing.

The proof of this lemma follows by integration of (20).

For the applications in the next section we need that the elements which are contained in the absorbing set satisfy a particular regularity. To this end we introduce the function space

$$
\mathcal{H}^s := \{u \in H : \|u\|_s^2 := \|A^{\frac{s}{2}}u\|_H^2 < \infty\}
$$

where $s \in \mathbb{R}$. The operator A^s is the s-th power of the positive and symmetric operator A. Note that these spaces are embedded in the Slobodeckij spaces H^s, $s > 0$. The norm of these spaces is denoted by $\| \cdot \|_{H^s}$. This norm can be found in Egorov and Shubin [7], Page 118. But we do not need this norm explicitly. We only mention that on $\mathcal{H}^s$ the norm $\|\cdot\|_s$ of H^s is equivalent to the norm of $\mathcal{H}^s$ for $0 < s$, see [8].

Our goal is it to show that $v(1, \omega, D)$ is a bounded set in $\mathcal{H}^s$ for some $s > 0$. This property causes the complete continuity of the mapping $v(1, \omega, \cdot)$. We now derive a differential inequality for $t\|v(t)\|_s^2$. By the chain rule we have

$$\frac{d}{dt}(t\|v(t)\|_s^2) = \|v(t)\|_s^2 + t\frac{d}{dt}\|v(t)\|_s^2.$$

Note that for the embedding constant c_s between $\mathcal{H}^s$ and V

$$\int_0^t \|v\|_s^2 ds \leq c_s^2 \int_0^t \|v\|_V^2 ds \quad \text{for } s \leq 1$$

such that the left hand side is bounded if the initial conditions v_0 are contained in a bounded set in H. The second term in the above formula can be expressed as followed:

$$t\frac{d}{dt}(A^{\frac{s}{2}}v, A^{\frac{s}{2}}v)_H = 2t(\frac{d}{dt}v, A^s v)_H = -2t(Au, A^s v)_H + 2t(F_1(v + \eta(\theta_t\omega)), A^s v)_H$$
$$+ 2t(F_2(v + \eta(\theta_t\omega)), A^s v)_H.$$

We have

$$(Av, A^s v)_H = \|A^{\frac{1}{2}+\frac{s}{2}}v\|_H = \|v\|_{1+s}^2.$$

Similar to the argument of [21] and the estimate for the existence of absorbing, and applying some embedding theorems, see Temam [18] Page 12 we have got

$$\|v\|_s^2 \leq C(t, \|v_0\|_H, \sup_{t\in[0,1]} \|\eta_1\|_{D(A^s)}), \text{ for } t \in [0,1].$$

By the results of [12] and [21], we know

$$\sup_{t\in[0,1]} \|\eta_1\|_{D(A^s)} \leq C(tr_{L^2}Q) < \infty, \text{ for } 0 < s < \frac{1}{4}.$$

By now, our estimates allow us to write down the main assertion with respect to the dissipativity of this section.

Theorem 4.3. *For the random dynamical system generated by (13), there exists a compact random set $B = \{B(\omega)\}_{\omega\in\Omega}$ which satisfies Definition 4.1.*

We define

$$(22) \qquad B(\omega) = \overline{\varphi(1, \theta_{-1}\omega, B(0, R(\theta_{-1}\omega)))} \subset \mathcal{H}^s, \quad 0 < s < \frac{1}{4}.$$

In particular, $\mathcal{H}^s$ is compactly embedded in H.

5. Random dynamics: Enstrophy and ergodicity

In this section we will apply the dissipativity result of the last section to analyse the dynamical behavior of the random dynamical system (1). However, it will be enough to analyse the transformed random dynamical system generated by (13). By the transformation (12), we have all these qualitative properties to the original

gravity current system (1).

We will consider the following dynamical behavior: random attractors, asymptotic evolution of enstrophy and mean-square norm of salinity profile, and ergodicity. Enstrophy is defined as one half of the mean-square integral of vorticity. Ergodicity implies that the time average for observables of the dynamical system approximates the statistical ensemble average, as long as the time interval is sufficiently long. We first consider random attractors. We recall the following basic concept; see, for instance, Flandoli and Schmalfuß [9].

Definition 5.1. *Let φ be a random dynamical system. A random set $A = \{A(\omega)\}_{\omega \in \Omega}$ consisting of compact nonempty sets $A(\omega)$ is called random global attractor if for any random bounded set D we have for the limit in probability*

$$(\mathbb{P}) \lim_{t \to \infty} \text{dist}_H(\varphi(t, \omega, D(\omega)), A(\theta_t \omega)) = 0$$

and

$$\varphi(t, \omega, A(\omega)) = A(\theta_t \omega)$$

any $t \geq 0$ and $\omega \in \Omega$.

The essential long-time behavior of a random system is captured by a random attractor. In the last section we showed that the dynamical system φ generated by (6) is dissipative which means that there exists a random set B satisfying (21). In addition, this set is compact. We now recall and adapt the following theorem from [9].

Theorem 5.2. *Let φ be a random dynamical dynamical system on the state space H which is a separable Banach space such that $x \to \varphi(t, \omega, x)$ is continuous. Suppose that B is a set ensuring the dissipativity given in Definition 4.1. In addition, B has a subexponential growth (see Definition 4.1) and is regular (compact). Then the dynamical system φ has a random attractor.*

This theorem can be applied to the random dynamical system φ generated by the stochastic differential equation (13). Indeed, all the assumptions are satisfied. The set B is defined in Theorem 4.3. Its subexponential growth follows from $B(\omega) \subset B(0, R(\omega))$ where the radius $R(\omega)$ has been introduced in the last section. Note that φ is a *continuous* random dynamical system; see Theorem 3.4. Thus φ has a random attractor. By the transformation (12), this is also true for the original gravity currents system (6).

Corollary 5.3. (**Random Attractor**) *The gravity currents system (6) has a random attractor.*

Now we consider random fixed point (stationary state) and ergodicity.

Definition 5.4. *A random variable $v^* : \Omega \to H$ is defined to be a random fixed point for a random dynamical system if*

$$\varphi(t, \omega, v^*(\omega)) = v^*(\theta_t \omega)$$

for $t \geq 0$ and $\omega \in \Omega$. A random fixed point v^* is called exponentially attracting if

$$\lim_{t \to \infty} \|\varphi(t, \omega, x) - v^*(\theta_t \omega)\|_H = 0$$

for any $x \in H$ and $\omega \in \Omega$.

Sufficient conditions for the existence of random fixed points are given in Schmalfuß [14]. We here formulate a simpler version of this theorem and it is appropriate for our system here.

Theorem 5.5. *(**Random Fixed Point Theorem***) Let φ be a random dynamical system and suppose that B is a forward invariant complete set. In addition, B has a subexponential growth, see Definition 4.1. Suppose that the following contraction conditions holds:*

$$(23) \qquad \sup_{v_1 \neq v_2 \in B(\omega)} \frac{\|\varphi(1, \omega, v_1) - \varphi(1, \omega, v_2)\|_H}{\|v_1 - v_2\|_H} \leq k(\omega)$$

where the expectation of $\log k$ denoted by $\mathbb{E} \log k < 0$. Then φ has a unique random fixed point in B which is exponentially attracting.

This theorem can be considered as a random version of the Banach fixed point theorem. The contraction condition is formulated in the mean for the right hand side of (23).

Theorem 5.6. *(**Unique Random Stationary State***) Assume that the salinity boundary flux data $\|F\|_{L_2}$, the Prandtl number Pr, and the trace of the covariance for the noise $\mathrm{tr}_{L_2} Q$ are sufficiently small. Then the random dynamical system generated by (6) has a unique exponentially attracting random stationary state.*

Note that if we take into account the horizontal and vertical diffusivities κ_H, κ_V in the Laplacian operator, the above "smallness" condition needs to be slightly modified.

Here we only give a short sketch of the proof. Let us suppose for a while that B is given by the ball $B(0, R)$ introduced in Lemma **??**. Suppose that the data in the assumption of the lemma are small and ν is large. Then it follows that $\mathbb{E}R$ is also small. To calculate the contraction condition we have to calculate $\|\varphi(1, \omega, v_1(\omega)) - \varphi(1, \omega, v_2(\omega))\|_H^2$ for arbitrary random variables $v_1, v_2 \in B$. By Lemma 3.2 we have that

$$\langle J(q_1, \psi_1) - J(q_1, \psi_1), q_1 - q_2 \rangle \leq c_{22} \|q_1 - q_2\|_{W_2^1}^2 + c_{23} \|q_1\|_{\overset{\circ}{W}_2^1(D)}^2 \|q_1 - q_2\|_{L_2}^2$$

where the constant c_{23} can be chosen sufficiently small if ν is large. On account of the fact that the other expressions allow similar estimates and that F_2 is linear we obtain:

$$\frac{d}{dt} \|\varphi(t, \omega, v_1(\omega)) - \varphi(t, \omega, v_2(\omega))\|_H^2$$
$$\leq (-\alpha' + c_{23} \|\varphi(t, \omega, v_2(\omega))\|_V^2) \|\varphi(t, \omega, v_1(\omega)) - \varphi(t, \omega, v_2(\omega))\|_H^2$$

for some positive α' depending on ν. From this inequality and the Gronwall lemma it follows that the contraction condition (23) is satisfied if

$$\mathbb{E} \sup_{u_2 \in B(\omega)} c_{23} \int_0^1 \|\varphi(t, \omega, v_2)\|_V^2 dt < \alpha'.$$

But by the energy inequality this property is satisfied if the $\mathbb{E}R$ and $\mathbb{E}\|z\|_V^2$ is sufficiently small which follows from the assumptions.

Let now B be the random set defined in (22). Since the set B introduced in (22) is absorbing any state the fixed point v^* is contained in this B. In addition v^* attracts *any* state from H and not only states from B.

The uniqueness of this random fixed point implies *ergodicity*. We will comment on this issue at the end of this section.

By the well-posedness Theorem 3.4, we know that the stochastic evoltuion equation (6) has a unique solution. The solution is a Markov process. We can define the associated Markov operators $\mathcal{T}(t)$ for $t \geq 0$, as discussed in [16, 15]. Moreover, $\{\mathcal{T}(t)\}_{t\geq0}$ forms a semigroup.

Let M^2 be the set of probability distributions μ with finite energy, i.e.,

$$\int_H \|u\|_H^2 d\mu(u) < \infty.$$

Then the distribution of the solution $u(t)$ (at time t) of the stochastic evolution equation (6) is given by

$$\mathcal{T}(t)\mu_0,$$

where the distribution μ_0 of the initial data is contained in M^2.

We note that the expectation of the solution $\|u(t)\|_H^2$ can be expressed in terms of this distribution $\mathcal{T}(t)\mu_0$:

$$\mathbb{E}\|u(t)\|_H^2 = \int_H \|u\|_H^2 d\mathcal{T}(t)\mu_0.$$

We can derive the following energy inequality in the mean, using our earlier estimates.

Corollary 5.7. *The dynamical quantity $u = (q, S)$ of the coupled atmospheric-ocean system satisfy the estimate*

$$\mathbb{E}\|u(t)\|_H^2 + \alpha\mathbb{E}\int_0^t \|u(\tau)\|_V^2 d\tau \leq \mathbb{E}\|u_0\|_H^2 + t\, c_{24} + t\,\mathrm{tr}_{L_2}Q,$$

where the positive constants c_{24} and α depend on physical data $F(x)$, Pr and Ra.

By the Gronwall inequality, we further obtain the following result about the asymptotic mean-square estimate. Recall that $\frac{1}{2}\mathbb{E}\int_D q^2 dD$ is the *enstrophy* for gravity currents. The following theorem gives asymptotic estimate for both enstrophy and mean-square salinity $\mathbb{E}\int_D S^2 dD$.

Theorem 5.8. *(Enstrophy and mean-square salinity) For the expectation of the dynamical quantity $u = (q, S)$ of the coupled atmospheric-ocean system, we have the asymptotic estimate*

$$\limsup_{t\to\infty}\mathbb{E}\|u(t)\|_H^2 = \limsup_{t\to\infty}\int_H \|u\|_H^2 d\mathcal{T}(t)\mu_0 \leq \frac{c_{24} + \mathrm{tr}_{L_2}Q}{c_{25}}$$

if the initial distribution μ_0 of the random initial condition $u_0(\omega)$ is contained in M^2. Here $c_{25} > 0$ depends only on physical data.

By the estimates of Corollary 5.7, we are able to use the well known Krylov-Bogolyubov procedure to conclude the existence of invariant measures of the Markov semigroup $\mathcal{T}(t)$.

Corollary 5.9. *The semigroup of Markov operators $\{\mathcal{T}(t)\}_{t\geq 0}$ possesses an invariant distribution μ_i in M^2:*

$$\mathcal{T}(t)\mu_i = \mu_i \quad for \ t \geq 0.$$

In fact, the limit points of

$$\left\{ \frac{1}{t} \int_0^t \mathcal{T}(\tau)\mu_0 d\tau \right\}_{t\geq 0}$$

for $t \to \infty$ are invariant distributions. The existence of such limit points follows from the estimate in Corollary 5.7.

In some situations, the invariant measure may be unique. For example, the unique random fixed point in Theorem 5.6 is defined by a random variable $u^*(\omega) = v^*(\omega) + \eta(\omega)$. This random variable corresponds to a unique invariant measure of the Markov semigroup $\mathcal{T}(t)$. More specifically, this unique invariant measure is the expectation of the Dirac measure with the random variable as the random mass point

$$\mu_i = \mathbb{E}\delta_{u^*(\omega)}.$$

Because the uniqueness of invariant measure implies ergodicity [13], we conclude that the gravity currents model (1) is ergodic under the suitable conditions in Theorem 5.6 for physical data and random noise. We reformulate it as the following ergodicity principle.

Theorem 5.10. (Ergodicity) *Assume that the salinity boundary flux data $\|F\|_{L_2}$, the Prandtl number Pr, and the trace of the covariance for the noise $\mathrm{tr}_{L_2}Q$ are sufficiently small. Then the gravity currents system (1) is ergodic, namely, for any observable of the gravity currents, its time average approximates the statistical ensemble average, as long as the time interval is sufficiently long.*

In the regime of ergodicity, the gravity currents system can be numerically simulated with (almost surely) one random sample. This is what we call ergodicity-based numerical simulation of stochastic systems, in contrast to sample-wise ("Monte Carlo") simulations.

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(V. P. Bongolan-Walsh and J. Duan) DEPARTMENT OF APPLIED MATHEMATICS, ILLINOIS INSTITUTE OF TECHNOLOGY, CHICAGO, IL 60616, USA
E-mail address: `bongven@iit.edu; duan@iit.edu`

(H. Gao) DEPARTMENT OF MATHEMATICS, NANJING NORMAL UNIVERSITY, NANJING 210097, CHINA
E-mail address, H. Gao: `gaohj@njnu.edu.cn`

(T. Özgökmen) RSMAS/MPO, UNIVERSITY OF MIAMI, MIAMI, FLORIDA, USA
E-mail address, T. Özgökmen: `tamay@rsmas.miami.edu`

(P. Fischer) ARGONNE NATIONAL LABORATORY, ARGONNE, ILLINOIS, USA
E-mail address, P. Fischer: `fischer@mcs.anl.gov`

(T. Iliescu) MATHEMATICS DEPARTMENT, VIRGINIA TECH, BLACKSBURG, VA 24061, USA
E-mail address, T.Iliescu: `iliescu@calvin.math.vt.edu`